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Gauge-Invariant and Covariant Operators in Gauge Theories

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We study Euler–Lagrange-type operators, not necessarily variational (i.e., derivable from a variational principle). We get a *master equation* which is well suited for many considerations. We obtain several results in gauge theories: the equivalence between gauge invariance and charge conservation, the relationship between general covariance and conservation laws associated with the energy-momentum tensor, and, in the case of free gauge fields, an interesting characterization of the associated variational operators.

1. INTRODUCTION

As is well known, the Lagrangian formulation of the classical theory of interaction between gauge and matter fields is the most convenient way of discussing gauge invariance, general covariance, and conservation laws. Moreover, the fiber bundle approach is almost unavoidable for a better understanding of the theory (Mayer, 1977; Konopleva and Popov, 1981).

In this paper we are concerned with Euler–Lagrange-type operators that are not necessarily variational. We establish a formula for their Lie derivative (master equation) which has a key role in our considerations. The formula has two terms and, roughly speaking, the vanishing of one term refers to the condition that operators are variational, while the vanishing of the other refers to the conservation laws. So it is clear that, for example, the gauge invariance of variational operators may be equivalent to charge conservation.

The paper is related to some earlier work on gauge theories (Mangiarotti and Modugno, 1985; Mangiarotti, 1986, 1987). The reader is urged to start with the Appendix, where we prove the master equation and discuss its relationship with variational operators. Concepts and results from jet spaces are needed. Let us note that, in general, the use of jet spaces, avoiding a

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great deal of unnecessary coordinate manipulations, makes the geometrical and physical meanings more transparent.

Section 2 introduces some preliminary notations, concepts, and results. Section 3 deals with gauge-invariant and generally covariant Lagrangian densities. The next two sections are devoted to the relations among gauge invariance, general covariance, and conservation laws. Finally, in the last section we consider the special case of free gauge fields and get an interesting characterization of the associated variational operators.

2. PRELIMINARIES

This section is a collection of basic notations, concepts, and results needed in subsequent considerations. The detailed proofs can be found in Mangiarotti and Modugno (1983, 1985) and Mangiarotti (1987).

2.1. The Configuration Bundle

Let $P \to M$ be a principal fiber bundle with structure Lie group G (Kobayashi and Nomizu, 1963). The base manifold M is the space-time manifold. A basic role is played by the quotient spaces $V_G P = VP/G \subset T_G P = TP/G$ which are vector bundles over M. The vertical bundle $V_G P \to M$ is a Lie algebra bundle (gauge algebra bundle) whose fibers are isomorphic to g, the Lie algebra of G.

A principal connection A on P (gauge potential) is then a splitting of the exact sequence of vector bundles over M,

$$O \to V_G P \to T_G P \overset{A}{\leftrightarrows} TM \to O \tag{2.1}$$

It follows that principal connections A are sections of an affine bundle $C \to M$ whose vector bundle is the tensor product bundle $VC = T^*M \otimes V_G P$.

Let $E \to M$ be a vector bundle associated with P (matter bundle) and let $L \subset \bigvee_2 TM \to M$ be the bundle of the Lorentz metrics (M must satisfy well-known topological conditions). Since we are concerned with the interaction among gravitational, gauge, and matter fields, our configuration bundle is $Q \to M$, where $Q = L \times_M K$ and $K = C \times_M E$.

In order to introduce local coordinates, we need to choose a local gauge $U \times G$ on P, over a coordinate neighborhood (U, x^λ) in M, and a basis (e_p) of the Lie algebra g, where $1 \le \lambda \le m = \dim M$ and $1 \le p \le n = \dim G$. Then we denote by c^r_{pq} the right structure constants of g with respect to the basis (e_p).

The induced standard coordinates on L, $T_G P$, and C are denoted by $(x^\lambda, g^{\alpha\beta})$, $(x^\lambda, \dot{x}^\lambda, \xi^p)$, and (x^λ, a^p_λ), respectively. If A is a principal connection, its local expression is

$$A = dx^\lambda \otimes (\partial_\lambda - A^p_\lambda e_p) \tag{2.2}$$

where A^p_λ are local functions on M. These functions are just the gauge potentials of A with respect to the gauge $U \times G$, chart (U, x^λ), and basis (e_p). About the coordinates on C, we use the convention $a^p_\lambda \circ A = -A^p_\lambda$.

A principal connection A induces a linear connection ∇^A in the vector bundle $V_G P \to M$. It is easily seen that the connection parameters for ∇^A are

$$\nabla^A e_q = -c^r_{pq} A^p_\lambda \, dx^\lambda \otimes e_r \tag{2.3}$$

Now let $\rho: g \to \mathrm{End}(F)$ be the Lie algebra representation associated with the matter bundle $E \to M$, F being its typical fiber. Let (e_i) be a basis of F, where $1 \le i \le l = \dim F$. Then we put $\rho(e_p) e_j = \rho^i_{pj} e_i$. The induced standard coordinates on E are denoted by (x^λ, φ^i). It is clear that ρ determines a morphism of Lie algebra bundles ρ: $V_G P \to E^* \otimes E$ whose local expression is just ρ^i_{pj}.

A principal connection A induces a linear connection ∇^A in the vector bundle $E \to M$. It is easily seen that the connection parameters for ∇^A are

$$\nabla^A e_j = \rho^i_{pj} A^p_\lambda \, dx^\lambda \otimes e_i \tag{2.4}$$

We have two canonical objects related to our configuration bundle $Q \to M$, namely the curvature 2-form $F: J_1 C \to \bigwedge^2 T^*M \otimes V_G P$ and the interaction 1-form γ: $C \times_M J_1 E \to T^*M \otimes E$. Using the induced coordinates $(x^\lambda, a^p_\lambda, a^p_{\mu,\lambda})$ and $(x^\lambda, \varphi^i, \varphi^i_\lambda)$ on $J_1 C$ and $J_1 E$, respectively, we find the local expressions of F and γ in the form

$$F^r_{\lambda\mu} = a^r_{\mu,\lambda} - a^r_{\lambda,\mu} - c^r_{pq} a^p_\lambda a^q_\mu \tag{2.5}$$

$$\gamma^i_\lambda = \varphi^i_\lambda - \rho^i_{pj} a^p_\lambda \varphi^j \tag{2.6}$$

The meaning of F and γ is clear. Let A: $M \to C$ be a principal connection and let ϕ: $M \to E$ be a matter field. Then $F \circ j_1 A = F_A$ is the curvature of A (field strength associated with A) and $\gamma \circ (A, j_1 \phi) = \nabla^A \phi$ is the covariant derivative of ϕ with respect to the linear connection ∇^A induced by A on $E \to M$.

2.2. Lagrangians and Operators

Let $\mathscr{L}$: $L \times_M J_1 K \to \bigwedge^m T^*M$ be a Lagrangian density. Locally we write $\mathscr{L} = L\omega$, where L is a local function on $L \times_M J_1 K$. The choice of this particular jet extension, in which we do not consider the jets of the Lorentz metrics, is motivated by the specific physical applications. If we fix a metric on M, this choice allows us to get automatically the metric energy-momentum tensor of the interaction between gauge and matter fields.

In the sequel we need the momentum map of $\mathscr{L}$, which is the direct sum of the two following objects:

$$(\pi, \chi): \quad L \underset{M}{\times} J_1 K \to \bigwedge^{m-1} T^*M \otimes (V^*C \oplus E^*) \tag{2.7}$$

$$\pi = \pi_p^{\lambda,\mu}\omega_\lambda \otimes \partial_\mu \otimes e^p, \qquad \pi_p^{\lambda,\mu} = \partial_p^{\lambda,\mu} L \tag{2.8}$$

$$\chi = \chi_i^\lambda \omega_\lambda \otimes e^i, \qquad \chi_i^\lambda = \partial_i^\lambda L \tag{2.9}$$

where $\omega_\lambda = \partial_\lambda \rfloor \omega$. Note that χ is an $(m-1)$-horizontal form valued on E^*. Using the representation ρ: $V_G P \to E^* \otimes E$, we can do a pullback of it, namely

$$\begin{aligned} j = \rho^*\chi\colon \quad & L \underset{M}{\times} J_1 K \to \overset{m-1}{\bigwedge} T^*M \otimes V_G^* P \\ j = j_p^\lambda \omega_\lambda \otimes e^p, \qquad & j_p^\lambda = \rho_{pj}^i \varphi^j \chi_i^\lambda \end{aligned} \tag{2.10}$$

As we shall see, this is just the current associated with the matter fields.

Coming to π, first of all note that $V^*C = TM \otimes V_G^* P$. Then it is clear that we cannot assert that π is an $(m-2)$-horizontal form valued on $V_G^* P$. As we shall see, this becomes true when $\mathscr{L}$ is gauge invariant, and this will be relevant in many considerations.

Let $E_{\mathscr{L}}$: $J_2 Q \to \bigwedge^m T^*M \otimes V^*Q$ be the Euler–Lagrange operator associated with $\mathscr{L}$ (see the Appendix). Since we have $V^*Q = \bigvee_2 T^*M \oplus TM \otimes V_G^* P \oplus E^*$, it follows from (A.4) that $E_{\mathscr{L}}$ is a direct sum, i.e., $E_{\mathscr{L}} = (\tau, \varepsilon, \eta)$, where

$$\tau\colon \quad L \underset{M}{\times} J_1 K \to \overset{m-1}{\bigwedge} T^*M \otimes T^*M, \qquad \tau = \tau_\alpha^\lambda \omega_\lambda \otimes dx^\alpha \tag{2.11}$$

$$\varepsilon\colon \quad J_1 L \underset{M}{\times} J_2 K \to \overset{m-1}{\bigwedge} T^*M \otimes V_G^* P, \qquad \varepsilon = \varepsilon_p^\mu \omega_\mu \otimes e^p \tag{2.12}$$

$$\eta\colon \quad J_1 L \underset{M}{\times} J_2 K \to \overset{m}{\bigwedge} T^*M \otimes E^*, \qquad \eta = \eta_i \omega \otimes e^i \tag{2.13}$$

The local expressions are

$$\tau_\alpha^\lambda = -2g^{\lambda\beta}\partial_{\alpha\beta} L \tag{2.14}$$

$$\varepsilon_p^\mu = \partial_p^\mu L - J_\lambda \pi_p^{\lambda,\mu} \tag{2.15}$$

$$\eta_i = \partial_i L - J_\lambda \chi_i^\lambda \tag{2.16}$$

Finally, we introduce the canonical energy-momentum tensor $T_{\mathscr{L}}$ associated with $\mathscr{L}$. This is an object as τ [see (2.11)], whose local expression is

$$T_\alpha^\lambda = L\delta_\alpha^\lambda - F_{\mu\alpha}^p \pi_p^{\lambda,\mu} - \gamma_\alpha^i \chi_i^\lambda \tag{2.17}$$

Its meaning will become clear later.

2.3. The Basic Representation

Let $\xi: M \to T_G P$ be a section. We know from (2.1) that ξ projects into a vector field on M, say $u: M \to TM$. A basic fact is that we can associate with ξ a vector field on Q, say $u_\xi: Q \to TQ$. This is just the infinitesimal version of the well-known fact that a principal automorphism on P induces an automorphism on C, E, and M (and hence on TM). The map $\xi \mapsto u_\xi$ is an $\mathbb{R}$-Lie algebras morphism. Its local expression is

$$\xi = u^\lambda \partial_\lambda + \xi^p e_p, \qquad u = u^\lambda \partial_\lambda$$
$$u_\xi = u^\lambda \partial_\lambda + (g^{\lambda\alpha}\partial_\lambda u^\beta + g^{\lambda\beta}\partial_\lambda u^\alpha)\partial_{\alpha\beta} + (\partial_\lambda \xi^r - a^r_\mu \partial_\lambda u^\mu + c^r_{pq} a^p_\lambda \xi^q)\partial^\lambda_r + \rho^i_{pj}\xi^p \varphi^j \partial_i \tag{2.18}$$

where u^λ and ξ^p are local functions on M.

Let us denote by the same symbol u_ξ the lift of the representation (2.18) on $L \times_M J_1 K$ according to (A1). We say that the Lagrangian density $\mathscr{L}: L \times_M J_1 K \to \bigwedge^m T^*M$ is *generally covariant* if we have $L_{u_\xi}\mathscr{L} = 0$ for each section $\xi: M \to T_G P$ (when the sections ξ are vertical, i.e., we have $\xi: M \to V_G P$, we speak of *gauge invariance*). Using (2.18) and (A1), we find that $\mathscr{L}$ is generally covariant iff the following conditions are satisfied:

$$\begin{aligned} \pi^{\lambda,\mu}_p + \pi^{\mu,\lambda}_p &= 0 \\ \pi^\lambda_q + c^r_{pq} a^p_\alpha \pi^{\lambda,\alpha}_r + \rho^i_{qj}\varphi^j \chi^\lambda_i &= 0 \\ c^r_{pq}(a^p_\alpha \partial^\alpha_r L + a^p_{\lambda,\alpha}\pi^{\lambda,\alpha}_r) + \rho^i_{qj}(\varphi^j \partial_i L + \varphi^j_\lambda \chi^\lambda_i) &= 0 \\ \tau^\lambda_\alpha &= T^\lambda_\alpha \\ \partial_\alpha L &= 0 \end{aligned} \tag{2.19}$$

where τ^λ_α and T^λ_α have been defined in (2.14) and (2.17), respectively. The first three conditions are just equivalent to the gauge invariance of $\mathscr{L}$.

Now let $E = (\tau, \varepsilon, \eta)$ be an Euler–Lagrange-type operator, i.e., τ, ε, and η are morphisms as shown in (2.11), (2.12), and (2.13), respectively. Moreover, let us denote by the same symbol u_ξ the lift of the representation (2.18) on $J_1 L \times_M J_2 K$ according to (A1). We say that E is *generally covariant* if we have $L_{\mu_\xi}E = 0$ for each section $\xi: M \to T_G P$ (when the sections ξ are vertical, we speak of *gauge invariance*). Since we have $L_{\mu_\xi}E = (L_{u_\xi}\tau, L_{u_\xi}\varepsilon, L_{u_\xi}\eta)$, it is clear that E is generally covariant (gauge invariant) iff τ, ε, and η are. From (A5) it follows that if $E = E_{\mathscr{L}}$, i.e., if E is the Euler–Lagrange operator associated with $\mathscr{L}$, then $E_{\mathscr{L}}$ is generally covariant (gauge invariant) if $\mathscr{L}$ is. In general, the converse is not true.

It is easily seen that the canonical energy-momentum tensor $T_{\mathscr{L}}$ associated with $\mathscr{L}$, as defined in (2.17), has the property

$$L_{u_\xi} T_{\mathscr{L}} = T_{L_{u_\xi}\mathscr{L}}$$

for each section $\xi: M \to T_G P$. It follows that $T_{\mathscr{L}}$ is generally covariant (gauge invariant) if $\mathscr{L}$ is.

As we have done for $\mathscr{L}$, we could write explicitly the conditions that E must satisfy to be generally covariant (gauge invariant), but we do not need them.

3. COVARIANT LAGRANGIAN DENSITIES

In this section we consider gauge-invariant and generally covariant Lagrangian densities. The aim is to obtain two identities which have a key role in subsequent considerations. We also consider some questions related to charge conservation.

3.1. Gauge-Invariant Lagrangians

As before, let $\mathscr{L}: L \times_M J_1 K \to \bigwedge^m T^*M$ be a Lagrangian density. Then we can interpret the conditions of gauge invariance, i.e., the first three conditions (2.19), in an interesting way (Mangiarotti, 1986, 1987). The first condition tells us that π is just an $(m-2)$-horizontal form defined on $L \times_M J_1 K$ and valued on $V_G^* P$. Then, using the jet shift exterior covariant differential induced by (2.3) (Mangiarotti, 1987), we can compute $\nabla \pi: J_1 L \times_M J_2 K \to \bigwedge^{m-1} T^*M \otimes V_G^* P$, getting

$$\nabla \pi = -(J_\lambda \pi_q^{\lambda,\mu} + c_{pq}^r a_\lambda^p \pi_r^{\mu,\lambda})\omega_\mu \otimes e^q \tag{3.1}$$

Note that the difference between (3.1) and the usual exterior covariant differential is that we have the formal derivatives instead of the partial ones. Hence, the form $\nabla \pi$ is lifted over $J_1 L \times_M J_2 K$.

It follows easily that the second condition (2.19) is equivalent to

$$\varepsilon = \nabla \pi - j \tag{3.2}$$

where $j = \rho^* \chi$ is the current given by (2.10).

A further computation for $\nabla^2 \pi$ yields

$$\begin{gathered} \nabla^2 \pi: \quad L \underset{M}{\times} J_1 K \to \overset{m}{\bigwedge} T^*M \otimes V_G^* P \\ \nabla^2 \pi = -c_{pq}^r (a_\lambda^p \partial_r^\lambda L + a_{\lambda,\alpha}^p \pi_r^{\lambda,\alpha} + \rho_{rj}^i a_\lambda^p \varphi^j \chi_i^\lambda)\omega \otimes e^q \end{gathered} \tag{3.3}$$

which will be used later. Note that $\nabla^2 \pi$ projects down onto the original space $L \times_M J_1 K$.

Finally, another computation shows that the third condition (2.19) is equivalent to

$$\nabla \varepsilon - \rho^* \eta = 0, \qquad J_\lambda \varepsilon_q^\lambda - c_{pq}^r a_\lambda^p \varepsilon_r^\lambda - \rho_{qj}^i \varphi^j \eta_i = 0 \tag{3.4}$$

where ∇ is again the jet differential as in (3.2) and $\rho^*\eta$ denotes the pullback as in (2.10). This is just, as we shall see, the charge conservation identity associated with the internal symmetries.

3.2. Generally Covariant Lagrangians

Suppose that $\mathscr{L}$ is gauge invariant and, moreover, suppose also that $\mathscr{L}$ satisfies the fourth condition (2.19), i.e., $\tau = T$. Then a direct computation yields

$$J_\lambda \tau^\lambda_\alpha - K^\beta_{\lambda\alpha}\tau^\lambda_\beta - F^r_{\mu\alpha}\varepsilon^\mu_r - \gamma^i_\alpha \eta_i = \partial_\alpha L \tag{3.5}$$

where $2K^\beta_{\lambda\alpha} = g^{\beta\mu}(\partial_\lambda g_{\mu\alpha} + \partial_\alpha g_{\mu\lambda} - \partial_\mu g_{\lambda\alpha})$ are the Christoffel symbols. It follows that the last condition (2.19) is equivalent to

$$\nabla\tau - F \rfloor \varepsilon - \gamma \rfloor \eta = 0 \tag{3.6}$$

which has an intrinsic meaning. Here ∇ denotes the jet shift exterior covariant differential induced by the Levi-Civita connection, as is clear from (3.5). Moreover, the symbol $\rfloor$ denotes obvious contraction. This is the energy-momentum tensor identity associated with the external symmetries.

Let us make some remarks about (3.6). Let $u: M \to TM$ be a vector field. Then we have

$$d_H(u \rfloor \tau) = \nabla u \rfloor \tau + u \rfloor \nabla\tau \tag{3.7}$$

where d_H is the formal exterior differential (in which we take the formal derivatives instead of the partial ones; see also the Appendix) and ∇ is the jet shift differential as in (3.6). We also have

$$\nabla u \rfloor \tau = -2(\partial_\lambda u^\alpha + K^\alpha_{\lambda\beta}u^\beta)g^{\lambda\gamma}\partial_{\gamma\alpha}L\omega = L_u g^{\alpha\beta}\partial_{\alpha\beta}L\omega \tag{3.8}$$

where we have used the definition (2.14) of τ.

Now suppose that we fix a metric on M and that u is a symmetry of this metric (i.e., a Killing vector field). Then from (3.6)-(3.8) it follows that $u \rfloor \tau$ is a conserved current.

3.3. On Charge Conservation

As is well known, the standard physical situation is that in which we have

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_M, \qquad \mathscr{L}_0 = \bar{\mathscr{L}}_0 \circ F, \qquad \mathscr{L}_M = \bar{\mathscr{L}}_M \circ \gamma \tag{3.9}$$

where $\bar{\mathscr{L}}_0$ and $\bar{\mathscr{L}}_M$ are Lagrangian densities defined on the spaces $L \times_M \bigwedge^2 T^*M \otimes V_G P$ and $L \times_M T^*M \otimes E$, respectively. In other words, this is the minimal coupling situation.

Let us put $\mathscr{L}_M = L_M\omega$, where L_M is a local function on $L \times_M C \times_M J_1E$. Then we have

$$\partial_p^\lambda L_M + \rho_{pj}^i \varphi^j \partial_i^\lambda L_M = 0 \tag{3.10}$$

i.e., $j_p^\lambda = -\partial_p^\lambda L_M$, as follows from (2.10). So the current j is just given by the first variation of the matter Lagrangian $\mathscr{L}_M$ with respect to the gauge potential.

From (3.9) it follows easily that both $\mathscr{L}_0$ and $\mathscr{L}_M$ (and hence $\mathscr{L}$) satisfy the first two conditions (2.19). Then, using (3.3), we easily see that the two following statements are equivalent

(i) Both $\mathscr{L}_0$ and $\mathscr{L}_M$ are gauge invariant.

(ii) $\mathscr{L}$ is gauge invariant and $\nabla^2\pi = 0$.

A gauge-invariant Lagrangian $\mathscr{L}_0$ is just a *free gauge field*, while the gauge-invariant Lagrangian $\mathscr{L}_M$ describes the interaction between gauge and matter fields without the mass terms. Note that $\varepsilon_0 = \nabla\pi$ is just the Euler–Lagrange operator associated with $\mathscr{L}_0$ and that $\nabla^2\pi = 0$ is just the charge conservation identity for free gauge fields. As we shall see later, if ε_0: $J_1L \times_M J_2K \to \bigwedge^{m-1} T^*M \otimes V_G^*P$ is a gauge-invariant Euler–Lagrange-type operator, then ε_0 is locally variational iff $\nabla\varepsilon_0 = 0$.

Now let us fix a Lorentz metric on M and let us consider any Lagrangian density $\mathscr{L}$: $J_1K \to \bigwedge^m T^*M$. It is interesting to see how, in this general situation, the condition $\nabla^2\pi = 0$ allows us to discuss, in a nice way, the conservation of charge in both of the two cases in which (i) one uses the dynamics of the gauge field without the equation of motion of the charged field at all and (ii) one uses the gauge invariance without using the dynamics of the gauge field at all.

(i) Suppose that $\mathscr{L}$ satisfies the first two conditions (2.19). Then from (3.2) we get $\nabla j = -\nabla\varepsilon$ and hence j yields conserved currents over the solutions of ε.

(ii) Suppose that $\mathscr{L}$ is gauge invariant. Then from (3.2) and (3.4) we get $\nabla j = -\rho^*\eta$ and hence j yields conserved currents over the solutions of η.

4. GAUGE INVARIANCE AND CHARGE CONSERVATION

In this section we consider locally variational operators of the Euler–Lagrange type. The main result is the relationship between the gauge invariance of such operators and the charge conservation identity. Moreover, we consider the special case in which the structure group G is Abelian or semisimple. Our results improve those of Horndeski (1981).

Theorem 4.1. Suppose that $E = (\tau, \varepsilon, \eta)$ is a locally variational Euler–Lagrange-type operator, that is, $\delta E = 0$, where δ is the variational operator introduced in the Appendix. Then the two following conditions are equivalent:

(i) E is gauge invariant or, equivalently, τ, ε, and η are.

(ii) $\nabla\varepsilon-\rho^*\eta$ projects down to the base space M, i.e., we get $\nabla\varepsilon-\rho^*\eta\colon M\to\bigwedge^m T^*M\otimes V_G^*P$.

Proof. Let $\xi\colon M\to V_GP$ be a vertical section. Then from (A12) we get

$$L_{u_\xi}E=\delta(u_\xi \rfloor E) \tag{4.1}$$

where, on the right, $u_\xi\colon K\to VK$ is the vertical field given by (2.18), that is,

$$u_\xi=(\partial_\lambda\xi^r+c^r_{pq}a^p_\lambda\xi^q)\partial^\lambda_r+\rho^i_{pj}\xi^p\varphi^j\partial_i \tag{4.2}$$

On the left-hand side of (4.1), u_ξ denotes the second lift of (4.2).

Using the identity

$$d_H(\xi \rfloor \varepsilon)=\nabla\xi \rfloor \varepsilon+\xi \rfloor \nabla\varepsilon \tag{4.3}$$

from (4.2) we get

$$u_\xi \rfloor E=\nabla\xi \rfloor \varepsilon+\xi \rfloor \rho^*\eta=d_H(\xi \rfloor \varepsilon)-\xi \rfloor (\nabla\varepsilon-\rho^*\eta) \tag{4.4}$$

Here ∇ denotes, as is clear, the jet shift differential induced by (2.3).

Since we have (see the Appendix) $\delta d_H(\xi \rfloor \varepsilon)=0$, using (4.4), from (4.1) we get

$$L_{u_\xi}E=-\delta[\xi \rfloor (\nabla\varepsilon-\rho^*\eta)] \tag{4.5}$$

Now let us denote by ψ_p the components of $\nabla\varepsilon-\rho^*\eta$ according to (3.4). Then, using (A4), we can write explicitly the right-hand side of (4.5). By requiring that it vanishes for each vertical section ξ, we get a set of equations that must be satisfied by the components ψ_p. The ones relative to the bundle $J_2L\to M$ are

$$\begin{aligned}\partial^{\lambda\mu}_{\alpha\beta}\psi_p+\partial^{\mu\lambda}_{\alpha\beta}\psi_p&=0\\ 2J_\lambda(\partial^{\lambda\mu}_{\alpha\beta}\psi_p)-\partial^{\mu}_{\alpha\beta}\psi_p&=0\\ \partial_{\alpha\beta}\psi_p-J_\lambda(\partial^{\lambda}_{\alpha\beta}\psi_p)+J_\lambda J_\mu(\partial^{\lambda\mu}_{\alpha\beta}\psi_p)&=0\end{aligned} \tag{4.6}$$

which imply that $\partial^{\lambda\mu}_{\alpha\beta}\psi_p=\partial^{\lambda}_{\alpha\beta}\psi_p=\partial_{\alpha\beta}\psi_p=0$. In other words, all the fiber derivatives of ψ_p with respect to the bundle $J_2L\to M$ vanish. From the other equations we get the same things concerning $J_3K\to M$. The result is proved. ■

Remark 4.2. Let $\xi\colon M\to T_GP$ be a section. Then it is easily seen that we have

$$L_{u_\xi}(\rho^*\eta)=\rho^*(L_{u_\xi}\eta),\qquad L_{u_\xi}\nabla\varepsilon=\nabla L_{u_\xi}\varepsilon \tag{4.7}$$

where u_ξ is the lift of (2.18) on the appropriate jet space.

Corollary 4.3. Suppose that the structure Lie group G is Abelian or semisimple. Then the following conditions are equivalent:

(i) E is gauge invariant or, equivalently, τ, ε, and η are.

(ii) $\nabla\varepsilon - \rho^*\eta = 0$.

Proof. Suppose that E is gauge invariant and let $\xi: M \to V_G P$ be a vertical section. Then from (4.7) we get $L_\xi(\nabla\varepsilon - \rho^*\eta) = 0$ for each ξ. This is equivalent to $c^r_{pq}\psi_r = 0$, $\forall p, q$. The result follows (Bourbaki, 1975). ∎

5. GENERALLY COVARIANT OPERATORS

In this section we consider locally variational and gauge-invariant operators of the Euler-Lagrange type. The main result is that these operators then are generally covariant iff the further identity $\nabla\tau - F \rfloor \varepsilon - \gamma \rfloor \eta = 0$ is satisfied.

Lemma 5.1. Let $\xi: M \to T_G P$ be a section and let A be a principal connection. Then the connection A splits the section ξ into a vertical part ξ_A and a horizontal part $u \rfloor A$ (the vector field $u: M \to TM$ is the projection of ξ) whose local expressions are

$$\xi_A = (\xi^p + A^p_\lambda u^\lambda)e_p, \qquad u \rfloor A = u^\lambda(\partial_\lambda - A^p_\lambda e_p) \tag{5.1}$$

as follows from (2.2). Let us denote by u_A the lift of $u \rfloor A$ over Q and by $\bar{u}_A: J_1 Q \to VQ$ its vertical part, as explained in the Appendix. Then we have

$$\begin{aligned}\bar{u}_A = &-(L_u g^{\alpha\beta})\partial_{\alpha\beta} - [u^\mu(a^r_{\mu,\lambda} + \partial_\lambda A^r_\mu + c^r_{pq} a^p_\lambda A^q_\mu)\\ &+ (a^r_\mu + A^r_\mu)\partial_\lambda u^\mu]\partial^\lambda_r - (\nabla^A_u \varphi^i)\partial_i\end{aligned} \tag{5.2}$$

Proof. The local expression of u_A is

$$\begin{aligned}u_A = u^\lambda\partial_\lambda &+ (g^{\lambda\alpha}\partial_\lambda u^\beta + g^{\lambda\beta}\partial_\lambda u^\alpha)\partial_{\alpha\beta} - [u^\mu\partial_\lambda A^r_\mu\\ &+ (A^r_\mu + a^r_\mu)\partial_\lambda u^\mu + c^r_{pq} a^p_\lambda A^q_\mu u^\mu]\partial^\lambda_r - \rho^i_{pj} A^p_\lambda u^\lambda \varphi^j \partial_i\end{aligned} \tag{5.3}$$

as follows from (2.18). Then we get (5.2) from (A2). ∎

Lemma 5.2. Let $E = (\tau, \varepsilon, \eta)$ be an Euler-Lagrange-type operator. Then we have

$$\begin{aligned}\bar{u}_A \rfloor E = u \rfloor (\nabla\tau - F \rfloor \varepsilon - \gamma \rfloor \eta) &+ \xi_u \rfloor (\nabla\varepsilon - \rho^*\eta)\\ &- d_H(u \rfloor \tau + \xi_u \rfloor \varepsilon)\end{aligned} \tag{5.4}$$

where $\xi_u: C \to V_G P$ is the morphism given by $\xi_u = u^\lambda(a^r_\lambda + A^r_\lambda)e_r$.

Proof. The result follows from (5.2) using (3.7) and (3.8). ■

Proposition 5.3. Suppose that E is locally variational and gauge invariant. Then we have

$$L_{u_\xi}E = \delta(\tilde{u}_A \rfloor E) = L_{u_A}E$$
$$= \delta[u \rfloor (\nabla\tau - F \rfloor \varepsilon - \gamma \rfloor \eta)] + u \rfloor (\nabla\varepsilon - \rho^*\eta) \qquad (5.5)$$

where u_ξ denotes the second lift of the section ξ: $M \to T_GP$. Note that (5.5) is independent of the connection A.

Proof. This follows from (5.4) using (A12) and the fact that

$$\delta d_H(u \rfloor \tau + \xi_u \rfloor \varepsilon) = 0 \quad ■$$

Remark 5.4. Let ξ: $M \to T_GP$ be a section. Then it is easily seen that we have

$$L_{u_\xi}\nabla\tau = \nabla L_{u_\xi}\tau, \qquad L_{u_\xi}F = 0, \qquad L_{u_\xi}\gamma = 0 \qquad (5.6)$$

Theorem 5.5. Suppose that E is locally variational. Then the two following conditions are equivalent:

(i) E is generally covariant or, equivalently, τ, ε, and η are.

(ii) $\nabla\varepsilon - \rho^*\eta = 0$, $\nabla\tau - F \rfloor \varepsilon - \gamma \rfloor \eta = 0$.

Proof. Suppose that E is generally covariant. Then it is easily seen that $\nabla\varepsilon - \rho^*\eta = 0$. From (5.5) it follows that $\nabla\tau - F \rfloor \varepsilon - \gamma \rfloor \eta$ projects down on the base space M. Using (5.6), we get that $\nabla\tau - F \rfloor \varepsilon - \gamma \rfloor \eta = 0$. The converse is trivial. ■

Remark 5.6. Let $E = (\tau, \varepsilon, \eta)$ be an Euler-Lagrange-type operator. Let A be a principal connection. We say that E is A-horizontally invariant if $L_{u_A}E = 0$. Clearly, if E is gauge invariant and horizontally invariant with respect to a certain connection A, then it is generally covariant.

Note that if $E = E_{\mathscr{L}}$ and $\mathscr{L}$ is horizontally invariant with respect to a certain connection A, then $E_{\mathscr{L}}$ is generally covariant if it is gauge invariant.

6. A PROPERTY OF FREE GAUGE FIELDS

In this section we consider a free gauge operator, i.e., an Euler-Lagrange-type operator defined on the bundle of connections, namely

$$\varepsilon_0: \quad J_2C \to \bigwedge^{m-1} T^*M \otimes V_G^*P$$

Our main result is the following. Suppose that ε_0 is gauge invariant. then $\delta\varepsilon_0 = 0$, i.e., ε_0 is locally variational iff $\nabla\varepsilon_0$ projects down on M, i.e., we get

$$\nabla\varepsilon_0: \quad M \to \bigwedge^{m} T^*M \otimes V_G^*P$$

Lemma 6.1. Let $u\colon C \to VC$ be a vertical field and let A be a principal connection. Let $x \in M$. Then there exists a vertical section $\xi\colon M \to V_G P$ such that

$$(u \circ j_2 A)(x) = (u_\xi \circ j_2 A)(x) \tag{6.1}$$

Here, on the left, the same symbol u denotes the second lift of the field on $J_2 C$.

Proof. Let us put $u = u^r_\lambda \partial^\lambda_r$, where u^r_λ are local functions on C. Moreover, let $\xi = \xi^r e_r$ be a vertical section. The vertical field $u_\xi\colon C \to VC$ associated with ξ is, according to (2.18),

$$u_\xi = (\nabla \xi)^r_\lambda \partial^\lambda_r, \qquad (\nabla \xi)^r_\lambda = \partial_\lambda \xi^r + c^r_{pq} a^p_\lambda \xi^q \tag{6.2}$$

Using (A8), we see that if we want to get (6.1), we must have

$$\begin{aligned}
\partial_\lambda \xi^r(x) &= c^r_{pq} A^p_\lambda(x) \xi^q(x) + u^r_\lambda(A(x)) \\
\partial^2_{\lambda\alpha} \xi^r(x) &= c^r_{pq} \partial_\alpha A^p_\lambda(x) \xi^q(x) + c^r_{pq} A^p_\lambda(x) \partial_\alpha \xi^q(x) \\
&\quad + J_\alpha u^r_\lambda(j_1 A(x)) \\
\partial^3_{\lambda\alpha\beta} \xi^r(x) &= c^r_{pq} \partial^2_{\alpha\beta} A^p_\lambda(x) \xi^q(x) + c^r_{pq}(\partial_\alpha A^p_\lambda(x) \partial_\beta \xi^q(x) + \partial_\beta A^p_\lambda(x) \partial_\alpha \xi^q(x)) \\
&\quad + c^r_{pq} A^p_\lambda(x) \partial^2_{\alpha\beta} \xi^q(x) + J_\alpha J_\beta u^r_\lambda(j_2 A(x))
\end{aligned} \tag{6.3}$$

Now (6.3) shows that, if we fix the value $\xi(x)$, then the derivatives of ξ in x are uniquely determined. Hence, the result follows. ■

Proposition 6.2. The variational condition (A6) is equivalent to

$$\lambda^*(u_\xi \rfloor d\varepsilon_0) = 0 \tag{6.4}$$

for each vertical section ξ. The operator λ^* is defined in (A3) and d denotes, as in (A6), the ordinary exterior differential. In other words, (6.4) is equivalent to $\delta\varepsilon_0 = 0$, which is just the necessary and sufficient condition for ε_0 to be derivable from a sheaf of local Lagrangian densities.

Proof. The result follows immediately from (A9) and (A10) using (6.1). ■

Theorem 6.3. Any two of the following statements imply the third:

(i) $\delta\varepsilon_0 = 0$.

(ii) ε_0 is gauge invariant.

(iii) $\nabla\varepsilon_0$ projects down to the base space M.

Proof. Let $\xi\colon M \to V_G P$ be a vertical section. Then the results follow easily from the master equation (A11),

$$L_{u_\xi} \varepsilon_0 = \delta(\nabla \xi \rfloor \varepsilon_0) + \lambda^*(u_\xi \rfloor d\varepsilon_0) \tag{6.5}$$

using (4.3). ■

Remark 6.4. Suppose that the structure Lie group G is Abelian or semisimple. Then the condition (iii) becomes $\nabla\varepsilon_0 = 0$.

APPENDIX

In this Appendix we introduce some notations, concepts, and results from jet spaces (Mangiarotti and Modugno, 1983), the minimum needed to establish the master equation, a basic result for our considerations. We also study its relationship with variational operators.

Let $Y \to M$ be a fiber bundle and let (x^λ, y^i) be fibered coordinates on Y, $1 \leq \lambda \leq m = \dim M$, $1 \leq i \leq l$, $l+m = \dim Y$. The induced coordinates on the first jet space $J_1 Y$ are denoted by $(x^\lambda, y^i, y^i_\lambda)$. The meaning is the following. Let $s: M \to Y$ be a (local) section and put $y^i \circ s = s^i$, which are local functions on M. Let $j_1 s: M \to J_1 Y$ be the first jet extension of the section s. Then we have $y^i_\lambda \circ j_1 s = \partial_\lambda s^i$, i.e., the partial derivatives of s^i with respect to the coordinates x^λ.

As usual, the coordinate fields associated with $(x^\lambda, y^i, y^i_\lambda)$ are denoted by ∂_λ, ∂_i, and ∂_i^λ, respectively. Note that ∂_i are local vertical fields on Y, i.e., local sections of the vector bundle $VY \to Y$, where $VY \subset TY$ is the vertical space to Y, TY being the tangent space to Y. Moreover, both the fields ∂_i and ∂_i^λ are local vertical fields on $J_1 Y$, i.e., local sections of the vector bundle $VJ_1 Y \to J_1 Y$. As we shall see, the vertical spaces are a main tool in our considerations.

All that we have said generalizes immediately to the higher-order jet spaces $J_k Y$, $k > 1$. For them the standard multi-index notation will be used. In any case, we will be only concerned with the lower jet extensions.

A basic operation on jet spaces is the following. Let $u: Y \to TY$ be a projectable vector field on Y. Locally we write $u = u^\lambda \partial_\lambda + u^i \partial_i$, where u^λ and u^i are local functions on M and Y, respectively. Then u can be lifted into a (projectable) vector field on $J_k Y$, say $\lambda u: J_k Y \to TJ_k Y$, whose local expression is

$$\begin{aligned} \lambda u &= u^\lambda \partial_\lambda + u^i \partial_i + u^i_\Lambda \partial_i^\Lambda \\ u^i_{\lambda+\Lambda} &= J_\lambda u^i_\Lambda - y^i_{\mu+\Lambda} \partial_\lambda u^\mu \end{aligned} \tag{A1}$$

where Λ is a multi-index and J_λ denotes the formal derivative with respect to x^λ. It is easily seen that $u \mapsto \lambda u$ is an $\mathbb{R}$-Lie algebra morphism.

Using the projections $\vartheta_k: J_{k+1} Y \times_M TJ_k Y \to VJ_k Y$ (Mangiarotti and Modugno, 1983), we can define a *vertical part* of λu, say $\overline{\lambda u}: J_{k+1} Y \to VJ_k Y$ whose local expression is

$$\overline{\lambda u} = \bar{u}^i \partial_i + J_\Lambda \bar{u}^i \partial_i^\Lambda, \qquad \bar{u}^i = u^i - y^i_\alpha u^\alpha \tag{A2}$$

Note that $\overline{\lambda u}$ is a vertical vector field iff u is vertical, i.e., $u^\lambda = 0$ for each $1 \leq \lambda \leq m$.

The lifting λ admits a dual operation (Bauderon, 1982) acting on vector densities in the following way. Let $\alpha: J_k Y \to \bigwedge^m T^*M \otimes V^*J_k Y$ be a vector density defined over J_kY and valued on the dual vertical space V^*J_kY. Locally we write $\alpha = \omega \otimes (\alpha_i dy^i + \alpha_i^\Lambda dy^i_\Lambda)$, where $\omega = dx^i \wedge \cdots \wedge dx^m$, while α_i and α_i^Λ are local functions on J_kY. Then α can be lifted into a vector density defined over $J_{2k}Y$ and valued on V^*Y, say

$$\lambda^*\alpha : J_{2k}Y \to \bigwedge^m T^*M \otimes V^*Y$$

whose local expression is

$$\lambda^*\alpha = (\alpha_i + (-1)^{|\Lambda|} J_\Lambda \alpha_i^\Lambda)\omega \otimes dy^i \tag{A3}$$

where $|\Lambda|$ denotes the length of Λ. The operator λ^* can be easily extended to vector densities α valued on arbitrary exterior products $\bigwedge^p V^*J_kY$. However, we do not need the explicit expression of this extension.

Now let $\mathscr{L}: J_kY \to \bigwedge^m TM$ be a Lagrangian density. Locally we write $\mathscr{L} = L\omega$, where L is a local function on J_kY. Let $d\mathscr{L}: J_kY \to \bigwedge^m T^*M \otimes V^*J_kY$ be the ordinary exterior differential of $\mathscr{L}$. Its local expression is

$$d\mathscr{L} = \omega \otimes (\partial_i L \, dy^i + \partial_i^\Lambda L \, dy^i_\Lambda)$$

Then the Euler–Lagrange operator associated with $\mathscr{L}$ is the morphism $E_\mathscr{L} = \lambda^* \, d\mathscr{L} = \delta\mathscr{L}: J_{2k}Y \to \bigwedge^m T^*M \otimes V^*Y$ whose local expression is

$$E_\mathscr{L} = (\partial_i L + (-1)^{|\Lambda|} J_\Lambda \partial_i^\Lambda L)\omega \otimes dy^i \tag{A4}$$

The *variational operator* $\delta = \lambda^* d$ has the property $\delta^2 = 0$ (Bauderon, 1982). It follows that $\delta E_\mathscr{L} = \delta^2 \mathscr{L} = 0$.

As is well known, if $\mathscr{L} = d_H \psi$, where ψ is an $(m-1)$-horizontal form, i.e., $\psi: J_{k-1}Y \to \bigwedge^{m-1} T^*M$, then we have $\delta d_H \psi = 0$. In other words, $E_\mathscr{L}$ vanishes identically. The operator d_H is the formal exterior differential (the lower index H recalls that d_H acts on horizontal forms). Locally we write $\psi = \psi^\lambda \omega_\lambda$, where ψ^λ are local functions on $J_{k-1}Y$ and

$$\omega_\lambda = \partial_\lambda \rfloor \omega = (-1)^{\lambda-1} dx^1 \wedge \cdots \wedge d\hat{u}^\lambda \wedge \cdots \wedge dx^m$$

Then the local expression of the Lagrangian density is $\mathscr{L} = J_\lambda \psi^\lambda \omega$.

An interesting formula is

$$L_{\lambda u} E_\mathscr{L} = L_{\lambda u} \delta \mathscr{L} = \delta L_{\lambda u} \mathscr{L} = E_{L_{\lambda u}\mathscr{L}} \tag{A5}$$

where $L_{\lambda u}$ denotes the Lie derivative with respect to the lift λu. Here and in what follows it is understood that the liftings must be taken with respect the appropriate prolongation of jet spaces (so we use the same symbol for them).

Let us define an *Euler–Lagrange-type operator* as a morphism of the following type $E: J_{2k}Y \to \bigwedge^m T^*M \otimes V^*Y$. The operator E is called *locally variational* if $\delta E = 0$. This is just the necessary and sufficient condition for E to be derivable from a sheaf of local Lagrangian densities (Bauderon, 1982).

We can show that $\delta E = 0$ is just equivalent to the condition

$$\lambda^*(\lambda u \rfloor dE) = 0 \tag{A6}$$

for each vertical field u: $Y \to VY$. The vertical fields can be replaced with projectable ones. Indeed, using (A2), we easily see that (A6) is equivalent to

$$\lambda^*(\overline{\lambda u} \rfloor dE) = 0 \tag{A7}$$

for each projectable vector field u: $Y \to TY$. In these conditions d denotes the ordinary exterior differential.

Let us consider the case of a second-order operator $E: J_2 Y \to \bigwedge^m T^*M \otimes V^*Y$. Locally we write $E = E_i \omega \otimes dy^i$, where E_i are local functions on $J_2 Y$. Let $u = u^i \partial_i$ be a vertical field on Y. Then, using (A1) or (A2), we see that its lift over $J_2 Y$ is

$$\lambda u = u^i \partial_i + J_\lambda u^i \partial_i^\lambda + J_\lambda J_\mu u^i \partial_i^{\lambda\mu} \tag{A8}$$

It follows that the variational condition (A6) can be written as

$$\begin{aligned}\lambda^*(\lambda u \rfloor dE) = {} & u^i(\partial_i E_j - \partial_j E_i + J_\lambda \partial_j^\lambda E_i - J_\lambda J_\mu \partial_j^{\lambda\mu} E_i)\omega \otimes dy^j \\ & + J_\lambda u^i (\partial_i^\lambda E_j + \partial_j^\lambda E_i - 2J_\mu \partial_j^{\lambda\mu} E_i)\omega \otimes dy^j \\ & + J_\lambda J_\mu u^i (\partial_i^{\lambda\mu} E_j - \partial_j^{\lambda\mu} E_i)\omega \otimes dy^j \end{aligned} \tag{A9}$$

Since the vertical field u is arbitrary, (A6) is equivalent to the following local conditions:

$$\begin{aligned} \partial_i^{\lambda\mu} E_j - \partial_j^{\lambda\mu} E_i &= 0 \\ \partial_i^\lambda E_j + \partial_j^\lambda E_i - 2J_\mu \partial_j^{\lambda\mu} E_i &= 0 \\ \partial_i E_j - \partial_j E_i + J_\lambda \partial_j^\lambda E_i - J_\lambda J_\mu \partial_j^{\lambda\mu} E_i &= 0 \end{aligned} \tag{A10}$$

Now let E be an Euler–Lagrange-type operator and let u be a projectable vector field on Y. Then we have

$$L_{\lambda u} E = \lambda^*(d\overline{\lambda u} \rfloor E + \overline{\lambda u} \rfloor dE) \tag{A11}$$

This is our *master equation*. It can be proved by a direct check using (A1)-(A3).

If the operator E is locally variational, from (A7) and (A11) it follows that

$$L_{\lambda u} E = \delta(\overline{\lambda u} \rfloor E) \tag{A12}$$

which holds for each projectable vector field u on Y.

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